

# Charge and spin fractionalization in strongly correlated topological insulators

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We construct an effective topological Landau-Ginzburg theory that describes general  $SU(N)$  incompressible quantum liquids of strongly correlated particles in two spatial dimensions. This theory characterizes the fractionalization of quasiparticle quantum numbers and statistics in relation to the topological ground-state symmetries, and generalizes the Chern-Simons, BF and hierarchical effective gauge theories to an arbitrary representation of the  $SU(N)$  symmetry group. Our main focus are fractional topological insulators with time-reversal symmetry, which are treated as generalizations of the  $SU(2)$  quantum Hall effect.

A two dimensional electron gas in a strong magnetic field can exhibit fractional quantum Hall effect (FQHE) and fluctuations that carry a fractionally quantized amount of the electron's elementary charge [1]. Similar fractionalization is also possible in topological insulators (TIs) with time-reversal (TR) symmetry [2–4]. Owing to a strong spin-orbit coupling, these materials exhibit protected gapless modes at their boundaries, but differ from the quantum Hall systems by their spectra and by lacking a conserved “charge” (spin). The latter prevents the observation of a quantized transverse conductivity and obscures the fact that the topological features of TIs are born in the  $SU(2)$  extension of the quantum Hall effect [5].

This paper is motivated by the growing interest in strongly correlated TIs that feature fractional excitations [6–12], and their promise to enable novel topological orders that have no analogue in quantum Hall states. We construct an effective field theory of generic 2D fractional TIs that can capture the spin non-conserving character of spin-orbit couplings. Fractional TIs can be obtained in various systems, and likely will be observed in the foreseeable future. One promising system are ultra-cold gases of bosonic atoms trapped in quasi 2D optical lattices. Superfluid to Mott insulator transitions can be easily arranged to remove any energy scales that could compete with the spin-orbit coupling [13]. At the same time, the recent development of artificial gauge fields for neutral atoms, created by stimulated Raman transitions between internal atomic states, has not only introduced the effective spin-orbit couplings [14], but also looks promising for reaching the fractional quantum Hall regime [15, 16].

Promising solid-state systems include spin-orbit materials with significant Coulomb interactions, and TI-superconductor heterostructures. The former materials feature frustration of the electrons' kinetic energy, either by lattice geometry [17] or gauge fields [18]. Low-energy excitations can carry spin in these systems, so their coupling to orbital motion can result with non-trivial topological properties. The heterostructure systems rely on a superconducting material to induce corre-

lations among electrons in the proximate quantum well made from a band-insulating TI. The proximity effect produces Cooper pairs in the TI, which in turn can form a superconducting state or a bosonic Mott-insulator [19]. A phase transition between these states in the quantum well can be tuned by a gate voltage, and the ensuing quantum critical point (QCP) is highly sensitive to relevant perturbations such as the spin-orbit coupling. Inter-orbital  $p$ -wave Cooper pairs are allowed in the TI, and their spin-orbit coupling can produce fractional TI ground states in the critical fan of the Mott QCP.

We wish to benefit from the achieved understanding of FQHE in the exploration of TIs, while staying true to their unique properties. Thus, we will first introduce an  $SU(2)$  gauge-theory description of TIs that views them as a generalization of the quantum Hall effect [5]. The  $SU(2)$  gauge symmetry is approximate and various spin non-conserving perturbations are allowed to remove it in realistic systems. Fortunately, this cannot jeopardize the topologically protected many-body quantum entanglement in the bulk and its manifestations such as fractional statistics that the  $SU(2)$  formalism will capture. We will first develop the  $SU(2)$  theory in band-insulating TIs, and use it to construct the field theory of fractional ground states that could be stabilized in the presence of appropriate interactions as surveyed above. While mainly focused on the theory construction, we will make physical predictions along the way on the possible fractional quantum numbers of excitations in relation to the symmetries and external magnetic and spin-orbit fields.

Consider the following minimal model of a band-insulating TI quantum well, where electrons have the two-state spin  $\sigma^z$  and orbital  $\tau^z$  quantum numbers:

$$H = \frac{(\mathbf{p} - \tau^z \mathcal{A})^2}{2m} + \Delta \tau^x - \mu \quad , \quad \mathcal{A} = -mv(\hat{\mathbf{z}} \times \mathbf{S}) \quad . \quad (1)$$

$\mathbf{S} = \frac{1}{2}\sigma^a \hat{\mathbf{r}}^a$  is the vector spin operator,  $\sigma^a$  and  $\tau^a$  are Pauli matrices, and we set  $\hbar = 1$ . This Hamiltonian describes an electron moving in the external  $SU(2)$  gauge field  $\mathcal{A}$  whose  $SU(2)$  charge is  $g = \tau^z$ . The given form of  $\mathcal{A}$  reproduces the Rashba spin-orbit coupling

$H_{\text{so}} = v \hat{\mathbf{z}}(\mathbf{S} \times \mathbf{p})\tau^z$ , and the orbital index  $\tau^z$  can be interpreted as the top or bottom surface of the quantum well. Tunneling  $\Delta$  between the two surfaces opens a band-gap in an otherwise massless Dirac spectrum. The spin-orbit coupling  $v$ , however, is responsible for all topological properties, which is clearly revealed by the non-zero “magnetic”  $\text{SU}(2)$  flux density:

$$\Phi^\mu = \epsilon^{\mu\nu\lambda}(\partial_\nu \mathcal{A}_\lambda - ig\mathcal{A}_\nu \mathcal{A}_\lambda) = \frac{1}{2}(mv)^2 \delta_{\mu 0} \tau^z \sigma^z \quad (2)$$

We have used Einstein’s notation for summation over the repeated indices and the (2+1)D Levi-Civita tensor  $\epsilon^{\mu\nu\lambda}$  ( $\mu, \nu \dots$  are space-time directions,  $\mu = 0$  indicates the temporal direction). This model naturally describes the “surface” spectrum of the  $\text{Bi}_2\text{Te}_3$ ,  $\text{Bi}_2\text{Se}_3$  and other TI quantum wells [20, 21].

Knowing the gauge-field description of the spin-orbit coupling, we can generalize the model (1) to any combination of spin-orbit couplings and external electromagnetic fields acting on particles with arbitrary spin  $S$ . The general  $\text{U}(1) \times \text{SU}(2)$  gauge field is the matrix  $\mathcal{A}_\mu = a_\mu + A_\mu^a \gamma^a$ , where  $\gamma^a$  for  $a \in \{x, y, z\}$  are the  $\text{SU}(2)$  generators in the spin  $S$  representation. The gauge flux

$$\Phi^\mu = \epsilon^{\mu\nu\lambda}(\partial_\nu \mathcal{A}_\lambda - ig\mathcal{A}_\nu \mathcal{A}_\lambda) = \phi_\mu + \Phi_\mu^a \gamma^a \quad (3)$$

couple to the  $\text{U}(1)$  charge  $e$  and  $\text{SU}(2)$  charge  $g$  in the Hamiltonian

$$H_0 = \frac{(\mathbf{p} - e\mathbf{a} - g\mathbf{A}^a \gamma^a)^2}{2m} - ea_0 - gA_0^a \gamma^a + \dots, \quad (4)$$

where the dots denote any other terms that operate in the orbital space and provide an energy gap. The temporal  $\Phi^0$  and spatial  $\Phi^i$  fluxes correspond to “magnetic” and  $90^\circ$ -rotated “electric” fields respectively. Defining the charge  $j_\mu$  and spin  $J_\mu^a$  operators,

$$\begin{aligned} j_0 &= 1, & j_i &= \frac{1}{m}(p_i - ea_i - gA_i^a \gamma^a) \\ J_0^a &= \gamma^a, & J_i^a &= \frac{1}{2}\{\gamma^a, j_i\} \end{aligned} \quad (5)$$

we obtain the following Heisenberg equation of motion from (4):

$$\frac{dj_i}{dt} = i[H_0, j_i] = \frac{e}{m}\epsilon_{i\nu\lambda}j_\nu\phi_\lambda + \frac{g}{2m}\epsilon_{i\nu\lambda}\{j_\nu, \Phi_\lambda^a \gamma^a\}. \quad (6)$$

If the gauge field  $\mathcal{A}_\mu$  and flux  $\Phi_\mu$  matrices commute with the Hamiltonian, the general periodic solution is

$$j_i(t) = c_i e^{i\omega t} e^{i\gamma^a \omega^a t} + \delta j_i, \quad (7)$$

where the first term describes cyclotron motion with  $\omega = e\phi_0/m$ ,  $\omega^a = g\Phi_0^a/m$  and  $c_y = ic_x$ . The second term  $\delta j_i$  is a constant drift current perpendicular to both “electric” and “magnetic” fields.

We will now concentrate on the drift current kinematics in quantum spin-Hall states whose global spin  $\text{U}(1)$

symmetry is reflected by  $[\Phi_\mu, H_0] = 0$ . Then, we are free to work in the representation  $\Phi_\mu = \phi_\mu + \Phi_\mu^z \gamma^z$ , where  $\gamma^z$  is the standard diagonal angular momentum matrix in the spin  $S$  representation. Setting  $dj_i/dt = 0$  in (6), we find:

$$\delta j_i = \left(e\phi_0 + g\Phi_0^z \gamma^z\right)^{-1} \left(e\phi_i + g\Phi_i^z \gamma^z\right). \quad (8)$$

What we will need, however, is a slightly different formula

$$\begin{aligned} \delta j_i &= \left[e^2 \phi_0^2 + 2eg\phi_0 \Phi_0 \gamma^z + g^2 \Phi_0^2 (\gamma^z)^2\right]^{-1} \\ &\quad \left[e^2 \phi_0 \phi_i + eg(\phi_0 \Phi_i + \phi_i \Phi_0) \gamma^z + g^2 \Phi_0 \Phi_i (\gamma^z)^2\right], \end{aligned} \quad (9)$$

which is obtained by inserting  $e\phi_0 + g\Phi_0^z \gamma^z$  and its inverse in the solution for  $\delta j_i$  in (8). It is not hard to see that (8) indirectly describes a quantum (spin) Hall effect.

We now turn to interacting systems and construct an effective topological field theory of spin  $S$  particles in quantum Hall states that produces the equations of motion (9) from its kinematics. Such a theory cannot be derived microscopically, but will be justified in the description of universal phenomena on the basis of symmetries, as is well established in the theory of critical phenomena. We will set  $e = g = 1$  for simplicity. Consider a Lagrangian density  $\mathcal{L} = \mathcal{L}_{\text{LG}} + \mathcal{L}_t$  in imaginary time:

$$\begin{aligned} \mathcal{L}_{\text{LG}} &= \frac{K}{2} \left| (\partial_\mu - i\mathcal{A}_\mu) \psi \right|^2 - t|\psi|^2 - t' \psi^\dagger \Phi_0 \psi \\ &\quad + u|\psi|^4 + v|\psi^\dagger \gamma^a \psi|^2 + v' |\psi^\dagger \Phi_0 \psi|^2 \end{aligned} \quad (10)$$

$$\begin{aligned} \mathcal{L}_t &= -\frac{i\eta}{2} \psi^\dagger \epsilon^{\mu\nu\lambda} \left[ (\partial_\mu - i\mathcal{A}_\mu) \left\{ \partial_\nu - i\mathcal{A}_\nu, \Phi_0 \right\} (\partial_\lambda - i\mathcal{A}_\lambda) \right. \\ &\quad \left. + \left\{ (\partial_\mu - i\mathcal{A}_\mu)(\partial_\nu - i\mathcal{A}_\nu)(\partial_\lambda - i\mathcal{A}_\lambda), \Phi_0 \right\} \right] \psi, \end{aligned}$$

where  $\mathcal{L}_{\text{LG}}$  is the Landau-Ginzburg Lagrangian of a spinor field  $\psi$  whose components are  $\psi_s = \sqrt{\rho_s} \exp(i\theta_s)$ ,  $s = -S \dots S$ .  $\mathcal{L}_t$  is the “minimal” topological term allowed by symmetries, which generalizes a Chern-Simons coupling in the language of spinors. A factor of  $\Phi_0$  is necessary in  $\mathcal{L}_t$  if the time-reversal symmetry is to be protected in the absence of a  $\text{U}(1)$  magnetic field ( $\phi_0 = 0$ ), and we use anti-commutators to symmetrize  $\mathcal{L}_t$  with respect to the location of  $\Phi_0$ . We will show that  $\eta = \frac{1}{4}$  is real and topologically quantized, making  $\mathcal{L}_t$  act similar to a Berry’s phase.

The equations of motion are expressed in terms of the currents  $j_{\text{p}\mu} = \epsilon^{\mu\nu\lambda} \partial_\nu \tilde{j}_{\text{v}\lambda}$  and  $J_{\text{p}\mu}^a = \epsilon^{\mu\nu\lambda} \partial_\nu \tilde{J}_{\text{v}\lambda}^a$ , where the gauge-dependent phase currents of the  $\psi$  fields are:

$$\begin{aligned} \tilde{j}_{\text{v}\mu} &= -\frac{i}{2} \left[ \psi^\dagger \Phi_0 (\partial_\mu \psi) - (\partial_\mu \psi^\dagger) \Phi_0 \psi \right] \\ \tilde{J}_{\text{v}\mu}^a &= -\frac{i}{2} \left[ \psi^\dagger \gamma^a \Phi_0 (\partial_\mu \psi) - (\partial_\mu \psi^\dagger) \Phi_0 \gamma^a \psi \right]. \end{aligned} \quad (11)$$

This definition guaranties that  $j_{\text{p}\mu}$  and  $J_{\text{p}\mu}^a$  are gauge-invariant and transform as charge ( $j_0 \rightarrow j_0$ ,  $j_i \rightarrow -j_i$ )

and spin ( $J_0^a \rightarrow -J_0^a$ ,  $J_i^a \rightarrow J_i^a$ ) currents under time-reversal,  $\psi_s \rightarrow \psi_{-s}^*$ ,  $\phi_0 \rightarrow -\phi_0$ . The equations of motion for drift currents follow from  $\epsilon^{\mu\nu\lambda}(\partial_\nu - i\mathcal{A}_\nu)(\partial_\lambda - i\mathcal{A}_\lambda) = \epsilon^{\mu\nu\lambda}\partial_\nu\partial_\lambda - i\Phi^\mu$  and:

$$\psi^\dagger \frac{\partial \mathcal{L}_t}{\partial \psi^\dagger} = -\frac{i\eta}{2} \psi^\dagger \{ \partial_\mu - i\mathcal{A}_\mu, \Phi_0 \} \\ \times \left( \epsilon^{\mu\nu\lambda} \partial_\nu \partial_\lambda - i\Phi^\mu \right) \psi + h.c. = 0. \quad (12)$$

Clearly, the field configurations that satisfy

$$\psi^\dagger \Phi_0 \left( \epsilon^{\mu\nu\lambda} \partial_\nu \partial_\lambda - i\mathcal{G}\Phi^\mu \right) \psi = 0 \quad (13)$$

also satisfy (12). We again assume the global spin U(1) symmetry and choose the representation  $\Phi_\mu^a = \Phi_\mu^z \delta_{az}$ . Defining  $\Gamma_n = \langle \psi^\dagger (\gamma^z)^n \psi \rangle$  at the action saddle-point in incompressible states, we find

$$j_{p\mu} = \phi_0 \phi_\mu \Gamma_0 + (\Phi_0^z \phi_\mu + \phi_0 \Phi_\mu^z) \Gamma_1 + \Phi_0^z \Phi_\mu^z \Gamma_2 \quad (14) \\ J_{p\mu}^z = \phi_0 \phi_\mu \Gamma_1 + (\Phi_0^z \phi_\mu + \phi_0 \Phi_\mu^z) \Gamma_2 + \Phi_0^z \Phi_\mu^z \Gamma_3.$$

If we now interpret (9) as a renormalized current  $\delta j_i = j_i/j_0$  that describes a single particle  $\delta j_0 = 1$ , then the many-body quantum average  $\langle j_i \rangle = \langle j_0 \delta j_i \rangle$  reproduces (14) after replacements  $\langle (\gamma^z)^n \rangle \rightarrow \Gamma_n$ ,  $\langle j_\mu \rangle \rightarrow j_{p\mu}$ . Analogous correspondence between equations of motion is found for spin currents by inserting a factor of  $\gamma^z$  in (13). Therefore,  $\mathcal{L}_t$  captures the kinematics of drift currents in the combined U(1) and SU(2) “electromagnetic” fields. Note that additional solutions allowed by (12) correspond to fluctuations that are suppressed in quantum Hall liquids.

The Landau-Ginzburg part of this theory applied to bosons (microscopically formulated on a lattice) describes superfluid and Mott insulator phases, whose transition is driven by phase  $\theta_s$  fluctuations of the spinor components. If the fluctuations respected the U(1) $^{2S+1}$  symmetry, the dynamics near the transition would be captured by  $2S+1$  copies of the quantum XY model. The superfluid-Mott transition can be viewed as the condensation of quantized vortices according to the duality transformation of this model. Particles are mobile and coherent in the superfluid state, while vortices are gapped and localized into a vortex lattice if their density is finite. A Mott insulator is a dual reflection of the superfluid where particles and vortices exchange their behavior.

We can qualitatively view quantum Hall states as “arrested” Mott transitions in which both particles and vortices are abundant and mobile, yet uncondensed and controlled by the cyclotron scales. Duality allows simultaneous mobility of both particles  $j_{p\mu}$  and vortices  $j_{v\mu}$  only if vortices are “attached” to particles to prevent relative motion. In a quantum Hall state we must imagine that superfluid correlations are not locally lost, but particles have begun localizing so their wavefunctions must acquire vorticity due to the external flux. When every

particle becomes a microscopic “cyclotron” vortex, it experiences a Magnus force (dual to the Lorentz force) from the residual local phase coherence, which is captured by the topological term  $\mathcal{L}_t$ .

Armed with this duality picture, we can envision characteristic field configurations  $\psi$  in quantum Hall states, which must contain a finite density of singularities in order to take advantage of  $\mathcal{L}_t$ . Amplitude fluctuations  $\rho_s = \langle \psi_s^\dagger \psi_s \rangle$  are frozen in an incompressible state, while phases may fluctuate freely as long as their winding along any infinitesimal space-time loop  $dC$  is quantized as an integer:

$$n_s = \frac{1}{2\pi} \oint_{dC} dl_\mu \partial_\mu \theta_s \in \mathbb{Z}. \quad (15)$$

If  $dA_\mu$  is the oriented surface element bounded by  $dC$ , then  $Q = j_{p\mu} dA_\mu$  is either the physical charge contained within area  $dA_0$  for  $dA_\mu = dA_0 \delta_{\mu,0}$ , or the charge pushed through a line segment  $dl_i$  in a time interval  $dt$  for  $dA_\mu = dl_i dt \delta_{\mu,i}$ . We can similarly extract the amount of spin  $S^z = J_{p\mu}^z dA_\mu$ , and use (11) to obtain:

$$Q = \oint_{dC} dl_\mu \tilde{j}_{v\mu} = \sum_{s=-S}^S 2\pi n_s (\phi_0 + s\Phi_0^z) \rho_s \quad (16) \\ S^z = \oint_{dC} dl_\mu \tilde{J}_{v\mu}^z = \sum_{s=-S}^S 2\pi n_s (\phi_0 + s\Phi_0^z) s \rho_s.$$

There are  $2S+1$  spinor phases  $\theta_s$  for spin- $S$  particles whose vorticities  $n_s$  must be quantized. We can characterize a topological ground-state by  $n_s = m_s(Q, S^z)$  vorticities attached to a microscopic particle with quantum numbers  $(Q, S^z)$ . But, only when  $m_s(Q, S^z) = m_s \delta_{s,S^z}$  we can solve (16) to find  $\rho_s$  that are independent of  $(Q, S^z)$ :  $\rho_s = [2\pi m_s (\phi_0 + s\Phi_0^z)]^{-1}$ . Therefore, a ground-state is defined by  $2S+1$  integers  $m_s$  (restricted by  $\rho_s > 0$ ) whose reciprocals generalize the concept of filling factors. Fractional excitations carry quantum numbers generated by all combinations of  $n_s \in \mathbb{Z}$  in (16):

$$\delta Q = \sum_s \frac{n_s}{m_s}, \quad \delta S^z = \sum_s \frac{n_s}{m_s} s, \quad (17)$$

and are related to the ground-state symmetries by (14). The ground-state charge and spin densities are:

$$j_{p0} = \sum_{s=-S}^S \frac{\phi_0 + s\Phi_0^z}{2\pi m_s}, \quad J_{p0}^z = \sum_{s=-S}^S \frac{s\phi_0 + s^2\Phi_0^z}{2\pi m_s}. \quad (18)$$

For spin  $S = \frac{1}{2}$  particles in magnetic field with the filling factor  $\nu = 2m_{+1/2}^{-1} = 2m_{-1/2}^{-1}$ , the absence of a spin-orbit coupling  $\Phi_0^z = 0$  yields the Laughlin sequence  $j_{p0} = \nu\phi_0/2\pi$ ,  $J_{p0}^z = 0$ , with elementary fractional excitations  $\delta Q = \nu/2$ ,  $\delta S^z = \pm\nu/4$ . We see that spin

must be fractionalized just like charge, effectively reducing  $\hbar$  by an integer because the  $S^a$  operators must obey the Lie algebra. A correlated TR-invariant TI in zero magnetic field  $\phi_0 = 0$  exhibits the same combined spin and charge fractionalization when  $m_{+1/2} = -m_{-1/2}$ . Charge and spin can be independently fractionalized for  $m_{+1/2} \neq \pm m_{-1/2}$ , but this generally corresponds to a broken TR-symmetry, even in the zero magnetic field.

Fractionalization is dynamically related to vortex winding numbers, but fractional statistics and topological orders are shaped by the topological term  $\mathcal{L}_t$ , which is sensitive only to vortex and monopole topological defects of the “gauge fields”  $b_{s\mu} = \partial_\mu \theta_s$ . Let us integrate by parts the left-most derivative of  $\mathcal{L}_t$  in (10) and write  $\mathcal{L}_t = \mathcal{L}'_t + \delta\mathcal{L}_t$ , where  $\delta\mathcal{L}_t$  is the total derivative of a field bilinear. By Gauss’ theorem,  $\delta\mathcal{L}_t$  picks monopoles  $\partial_\mu(\epsilon^{\mu\nu\lambda}\partial_\nu b_{s\lambda}) \neq 0$ , which can exist only at the system boundaries because  $b_{s\mu}$  are phase gradients. The bulk  $\mathcal{L}'_t$  is sensitive to vortex singularities and yields Chern-Simons (CS) and “background field” (BF) effective theories in incompressible states with emergent  $U(1)^n$  symmetry.

Consider the vicinity of a vortex  $\psi_s = \sqrt{\rho_s} \exp(i\theta_s)$ , where  $\theta_s$  winds  $m_s$  times about the flux tube at  $\mathbf{r}_0$ . We define phase gauge fields  $b_{s\mu} = \partial_\mu \theta_s$  and organize them into a diagonal matrix  $B_\mu = \text{diag}(b_{s\mu})$  whose flux is  $\Phi_B^\mu = \text{diag}(\phi_s^\mu)$ . As  $\rho_s \approx \text{const}$ ,  $\phi_s^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu b_{s\lambda}$  vanishes in the plane perpendicular to the tube, except at  $\mathbf{r} = \mathbf{r}_0$ . We assume that important field configurations have a finite density of vortex defects. The phase  $\theta_s$  fluctuation kinematics is captured by:

$$\begin{aligned} \mathcal{L}'_t &= \frac{i\eta}{2} \left[ (\partial_\mu - i\mathcal{A}_\mu)\psi \right]^\dagger \Phi_0 \left[ (\epsilon^{\mu\nu\lambda}\partial_\nu \partial_\lambda - i\Phi^\mu)\psi \right] + \dots \\ &= \frac{i\eta}{2} \text{tr} \left[ (B_\mu - \mathcal{A}_\mu)\Phi_0(\Phi_B^\mu - \Phi^\mu)(\psi\psi^\dagger) \right] + \dots \end{aligned} \quad (19)$$

The other symmetrized terms in  $\mathcal{L}'_t$  denoted by ellipses yield the same kind of final result. In the  $S = 0$  representation we immediately recover the CS theory [1] by fixing the topological coupling to  $\eta = \frac{1}{4}$ , substituting the quantized density  $\rho = [2\pi m\phi_0]^{-1}$ , and defining  $b_\mu \rightarrow mc_\mu$  to associate one flux quantum of  $c_\mu$  with  $m$  windings of  $\theta$ , that is a microscopic particle. The current becomes  $j_{p\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu c_\lambda$ , and

$$\mathcal{L}'_t \rightarrow \mathcal{L}_{\text{CS}} = -i \left[ -\frac{m}{4\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu c_\lambda + j_{p\mu}a_\mu \right]. \quad (20)$$

In the  $S = \frac{1}{2}$  representation of the  $U(1)^2$  symmetry (conserved charge and  $S^z$ ),  $\psi\psi^\dagger$  is a matrix whose off-diagonal elements  $\sqrt{\rho_s\rho_{s'}}\exp[i(\theta_s - \theta_{s'})]$ ,  $s \neq s'$  do not matter under the trace in (19) when all other matrices are diagonal. Assuming a paramagnetic TR-invariant quantum Hall liquid  $m_{\pm 1/2} = \pm m$ , we can write  $B_\mu = \frac{m}{2}(c_\mu^s + c_\mu^c\sigma^z)$  to obtain the BF theory [9, 12]:

$$\mathcal{L}'_t \rightarrow \mathcal{L}_{\text{BF}} = -i \left[ -\frac{m}{4\pi}\epsilon^{\mu\nu\lambda}c_\mu^c\partial_\nu c_\lambda^s + j_{p\mu}a_\mu + J_{p\mu}^z A_\mu^z \right] \quad (21)$$

in the  $\mathcal{A}_0 = -\epsilon_{ij}x_i\Phi_j$ ,  $\mathcal{A}_i = -\frac{1}{2}\epsilon_{ij}x_j\Phi_0$  gauge, with currents  $j_{p\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu c_\lambda^c$  and  $J_{p\mu}^z = \frac{1}{4\pi}\epsilon^{\mu\nu\lambda}\partial_\nu c_\lambda^s$ . The transverse spin conductivity is  $\sigma_{xy}^s = J_{pi}^z/\phi_i = (2\pi m)^{-1}$ .

Hierarchical states are obtained by enlarging the spinor symmetry group to  $[U(1) \times SU(2)]^n$  and using a general linear relationship between the particle currents and the curls of vortex currents (11). This formally captures an arbitrary set of emergent low-energy modes that participate in the quantum Hall liquid. If the emergent low-energy symmetry is limited to  $U(1)^{2n}$ , then the effective theory takes a CS or BF form [22].

The main benefit of the effective topological Lagrangian  $\mathcal{L}_t$  in (10) is that it is not limited to quantum spin-Hall states like the CS theory, but can also describe incompressible quantum liquids shaped by the  $SU(2)$  gauge fields like (1) that do not conserve spin. Such fractional topological liquids without an analogue in quantum Hall states may be possible to obtain in the Rashba spin-orbit-coupled TIs, and will be studied elsewhere. Their topological orders are guaranteed to be captured by  $\mathcal{L}_t$  due to its large symmetry, even though we could justify  $\mathcal{L}_t$  via equations of motion only by scrutinizing the special cases of quantum spin-Hall states whose symmetry is restricted. Even more generally,  $\mathcal{L}_t$  can describe topological orders in any representation of any gauge symmetry group. Note, however, that the written  $\mathcal{L}_{\text{LG}}$  in (10) can be justified only in quantum Hall states.

Realistic systems have “perturbations” that violate the  $SU(2)$  gauge symmetry of any finite- $S$  representation. This is not a priori detrimental to topological order, but may gap out edge states [7] and spoil the spin-Hall effect. Also, the quantum numbers of fractional quasiparticles depend on the conserved quantities. Charge is always conserved and can be fractionalized, while “fractional spin” has to be understood in general as a quasiparticle’s degree of freedom derived from the electron’s spin that transforms non-trivially under TR. It becomes a quantum number only if  $S^z$  is conserved, or if no perturbations spoil the symmetries of the ideal Rashba spin-orbit-coupled model (1).

In conclusion, the topological term of (10) describes topological orders of general incompressible quantum liquids shaped by externally applied magnetic and spin-orbit fields in the continuum limit. Not all such liquids are quantum Hall states in non-Abelian “magnetic” fields.

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